Trees, Dyck paths, and homological dimensions of Nakayama algebras

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Young Researchers Conference on Combinatorial Synergies

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Linear Nakayama Algebras

Start with a directed graph Q:

$$\mathbf{0} \longrightarrow \mathbf{1} \longrightarrow \mathbf{2} \cdots\cdots\rightarrow \mathbf{n-2} \longrightarrow \mathbf{n-1}$$

Example:

$$\mathbf{0} \xrightarrow{\mathbf{a}} \mathbf{1} \xrightarrow{\mathbf{b}} \mathbf{2} \xrightarrow{\mathbf{c}} \mathbf{3} \xrightarrow{\mathbf{d}} \mathbf{4}$$

algebraically closed

Fix a field k. The path algebra kQ has

- basis = all paths in Q (including length-0 paths e_i)
- multiplication = concatenation of paths

Paths in our example:

$e_0,\ e_1,\ e_2,\ e_3,\ e_4$ (length 0)

$a,\ b,\ c,\ d$ (length 1)

$ab,\ bc,\ cd$ (length 2)

$abc,\ bcd$ (length 3)

$abcd$ (length 4)

Linear Nakayama Algebras

Example: Q is

$$\mathbf{0} \xrightarrow{\mathbf{a}} \mathbf{1} \xrightarrow{\mathbf{b}} \mathbf{2} \xrightarrow{\mathbf{c}} \mathbf{3} \xrightarrow{\mathbf{d}} \mathbf{4}$$

Example: in $A = kQ$,

$$(2a + b + e_2)(bc + c)$$

$$= 2abc + 2\underbrace{a \cdot c}_{=0} + \underbrace{b \cdot bc}_{=0}$$

$$+ bc + \underbrace{e_2 \cdot bc}_{=0} + e_2 \cdot c$$

(b *does not start at* 2) (c *starts at* 2)

$$= abc + bc + c$$

Example: Q is

$$\mathbf{0} \xrightarrow{\mathbf{a}} \mathbf{1} \xrightarrow{\mathbf{b}} \mathbf{2} \xrightarrow{\mathbf{c}} \mathbf{3} \xrightarrow{\mathbf{d}} \mathbf{4}$$

Quotienting kQ by an ideal I kills paths.

Example: $I = \langle ab, bcd \rangle$, $A = kQ/I$

e_0, e_1, e_2, e_3, e_4	(length 0)
a, b, c, d	(length 1)
ab, bc, cd	(length 2)
abc, bcd	(length 3)
$abcd$	(length 4)

Linear Nakayama Algebras

linear Nakayama algebra

$$\mathsf{A} = kQ/I, \quad I = \langle ab,\ bcd \rangle$$

Example: Q is

$$\mathbf{0} \xrightarrow{\mathbf{a}} \mathbf{1} \xrightarrow{\mathbf{b}} \mathbf{2} \xrightarrow{\mathbf{c}} \mathbf{3} \xrightarrow{\mathbf{d}} \mathbf{4}$$

Example: $(2a + b + e_2)(bc + c)$

$$= \underbrace{2a \cdot bc}_{\in I} + \underbrace{2a \cdot c}_{=0} + \underbrace{b \cdot bc}_{=0}$$

$$+\, bc + \underbrace{e_2 \cdot bc}_{=0} + e_2 \cdot c$$

(c *starts at* 2)

(*b does not start at* 2)

$$= bc + c$$

Linear Nakayama Algebras

linear Nakayama algebra

$$\mathsf{A} = kQ/I, \quad I = \langle ab,\ bcd \rangle$$

Example: Q is

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$$+ bc + \underbrace{e_2 \cdot bc}_{=0} + e_2 \cdot c$$

(c *starts at* 2)

(b *does not start at* 2)

$$= bc + c$$

Linear Nakayama algebras are very nice combinatorial objects: the entire representation theory is encoded in the combinatorics of surviving paths.

Indecomposable modules

$A = kQ/I, \quad I = \langle ab \rangle$

$$\mathbf{0} \xrightarrow{a} \mathbf{1} \xrightarrow{b} \mathbf{2} \xrightarrow{c} \mathbf{3}$$

An indecomposable A-module $M(i,k)$ consists of k-linear combinations of surviving paths starting at vertex i and of length $< k$.
A acts by path concatenation.

All 8 indecomposables of A:

$M(0,1) \cong \langle e_0 \rangle$

$M(0,2) \cong \langle e_0,\ a \rangle$

$M(1,1) \cong \langle e_1 \rangle$

$M(1,2) \cong \langle e_1,\ b \rangle$

$M(1,3) \cong \langle e_1,\ b,\ bc \rangle$

$M(2,1) \cong \langle e_2 \rangle$

$M(2,2) \cong \langle e_2,\ c \rangle$

$M(3,1) \cong \langle e_3 \rangle$

The Auslander–Reiten quiver

We can record algebraic information about all the A-modules in the form of a directed graph called the Auslander–Reiten quiver:

- Vertices: indecomposable A-modules
- Arrows: irreducible maps — atomic non-invertible morphisms between indecomposables (that is, those that do not factor through a split mono or epi)

The Auslander–Reiten quiver completely determines A up to Morita equivalence.[1]

[1] this holds for representation-finite algebras in general (those where the Auslander–Reiten quiver has finitely many vertices) if we also specify the mesh relations

The Auslander–Reiten quiver

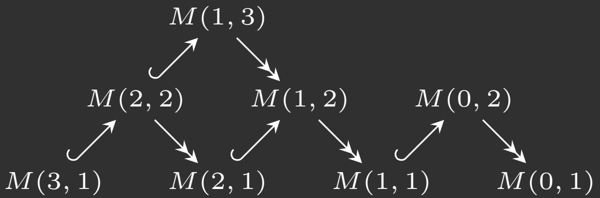
$A = kQ/I, \quad I = \langle ab \rangle$

$$\mathbf{0} \xrightarrow{a} \mathbf{1} \xrightarrow{b} \mathbf{2} \xrightarrow{c} \mathbf{3}$$

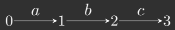
$M(i,k)$: indecomposable A-module

$\hookrightarrow$: irreducible inclusion

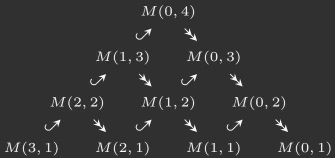
$\twoheadrightarrow$: irreducible projection

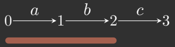


All linear Nakayama algebras on A_4

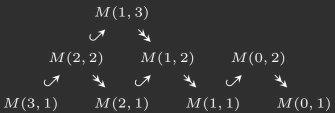


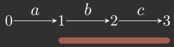
$$I = 0$$



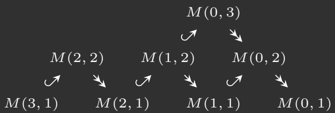


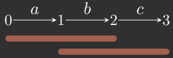
$$I = \langle ab \rangle$$



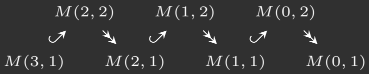


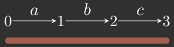
$$I = \langle bc \rangle$$



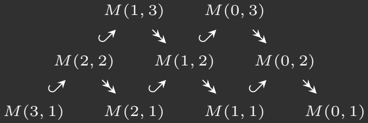


$$I = \langle ab, bc \rangle$$





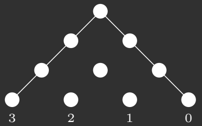
$$I = \langle abc \rangle$$



All linear Nakayama algebras on A_4

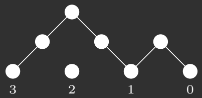
$$0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{c} 3$$

$$I = 0$$



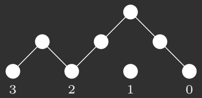
$$0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{c} 3$$

$$I = \langle ab \rangle$$



$$0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{c} 3$$

$$I = \langle bc \rangle$$



$$0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{c} 3$$

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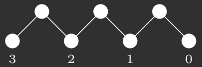

$$0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{c} 3$$

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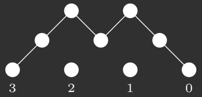


The AR quiver and Dyck paths

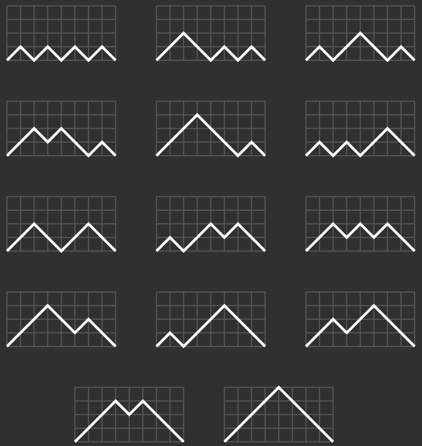


Figure: Dyck paths of semilength 4

A Dyck path of semilength n: a lattice path from $(0,0)$ to $(2n,0)$ using steps $\nearrow$ and $\searrow$, never going below the x-axis.

Bijection

Linear Nakayama algebras on n vertices

$\updownarrow$

Dyck paths of semilength $n-1$

Global dimension

A-module

- $\operatorname{gldim} A = \sup_{M} \operatorname{pdim}(M)$

Theorem (Klász–Marczinzik–Mellit–Rubey–Stump '25)

The statistic gldim *on linear Nakayama algebras on* n *vertices is equidistributed with the height of Dyck paths of semilength* $n-1$*:*

$$\#\{A : \operatorname{gldim} A = k\} = \#\{\textit{Dyck paths of semilength } n : \operatorname{ht} = k\}.$$

Projective resolutions

- A free module is a direct sum of copies of A (no relations)

- A projective module is a **direct summand** of a free module alt: has the lifting property

- A minimal projective resolution of M is an exact sequence

$$\begin{array}{ccccccccccccccccc}
0 & \hookrightarrow & P_m & \to & \cdots & \to & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \twoheadrightarrow & M & \to & 0 \\
 & & & & & & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & & & & \\
 & & & & & & & \Omega^3 & & \Omega^2 & & \Omega^1 & & & & &
\end{array}$$

with each P_i projective and "minimal"

- The projective dimension $\operatorname{pdim} M$ is the length of such a resolution

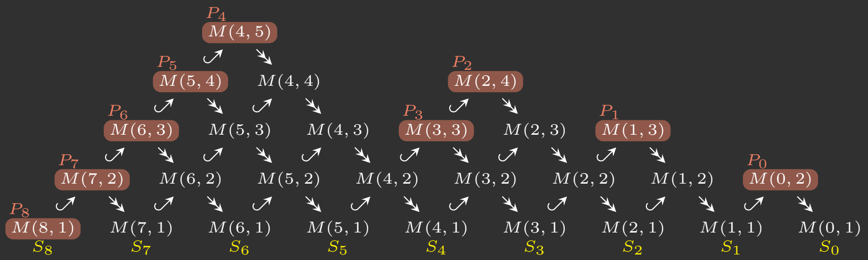
$\operatorname{pdim} M = 0 \iff M$ is projective ("best" case)

- $\operatorname{gldim} A = \sup_M \operatorname{pdim} M$ measures the **worst case**

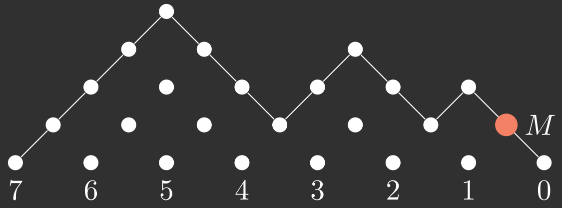
Projective modules

$P_i = M(i, c_i)$ has basis all paths starting at vertex i

orange = projective



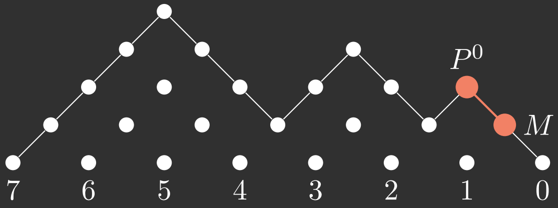
Projective resolution



$$0 \hookrightarrow \underset{P^3}{M(5,3)} \longrightarrow \underset{P^2}{M(3,5)} \longrightarrow \underset{P^1}{M(2,3)} \longrightarrow \underset{P^0}{M(0,3)} \twoheadrightarrow M(0,2) \longrightarrow 0$$

$$\underset{\Omega^3}{M(5,3)} \qquad \underset{\Omega^2}{M(3,2)} \qquad \underset{\Omega^1}{M(2,1)}$$

Projective resolution



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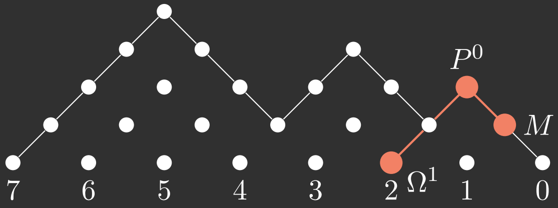
Projective resolution



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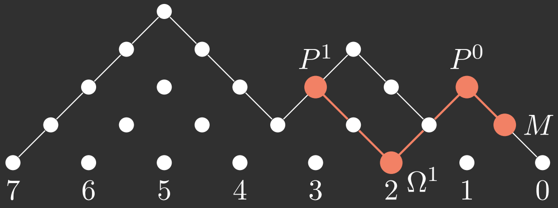
Projective resolution



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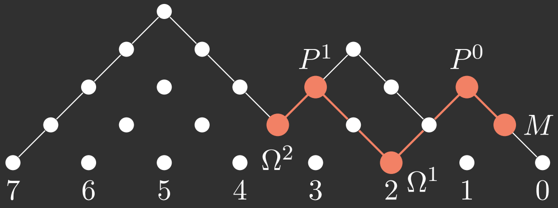
$$\Omega^3 = M(5,3), \quad \Omega^2 = M(3,2), \quad \Omega^1 = M(2,1)$$

Projective resolution



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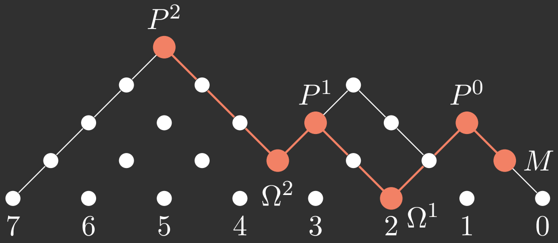
Projective resolution



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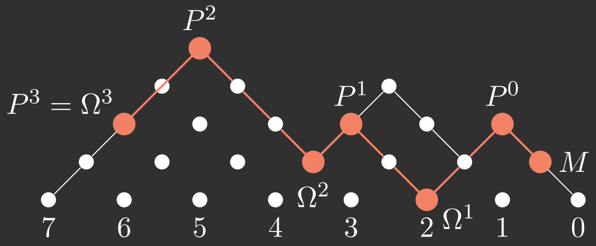
Projective resolution



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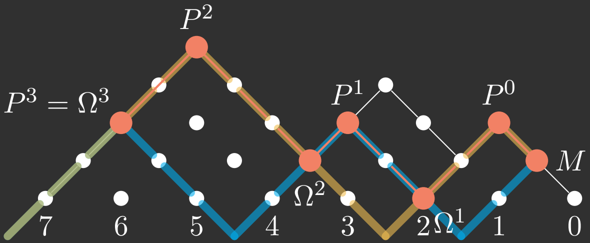
$$M(5,3) \twoheadrightarrow \underset{\Omega^3}{M(5,3)} \hookrightarrow M(3,5) \twoheadrightarrow \underset{\Omega^2}{M(3,2)} \hookrightarrow M(2,3) \twoheadrightarrow \underset{\Omega^1}{M(2,1)} \hookrightarrow M(0,3)$$

Minimal projective resolution



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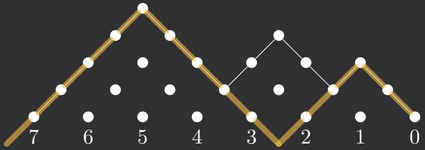
$$\underset{\Omega^3}{M(5,3)} \qquad \underset{\Omega^2}{M(3,2)} \qquad \underset{\Omega^1}{M(2,1)}$$

$M(5,3) \twoheadrightarrow M(5,3) \hookrightarrow M(3,5)$, $M(3,5) \twoheadrightarrow M(3,2) \hookrightarrow M(2,3)$, $M(2,3) \twoheadrightarrow M(2,1) \hookrightarrow M(0,3)$

Toolbox: resolution tree



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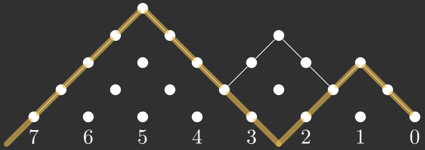

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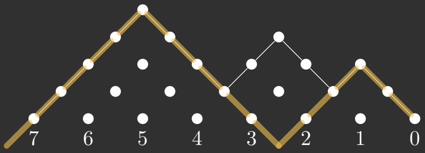

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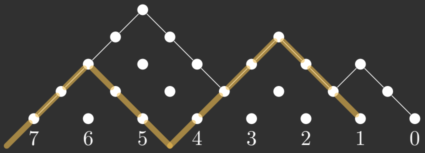

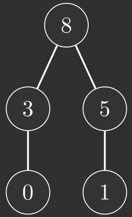

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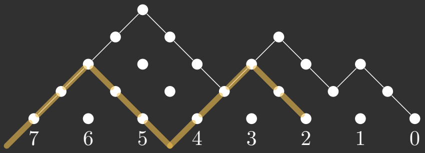

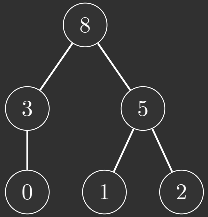

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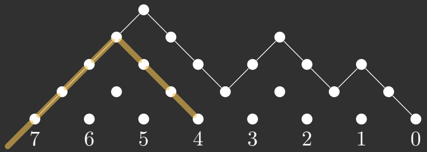

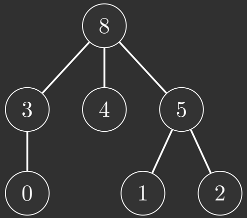

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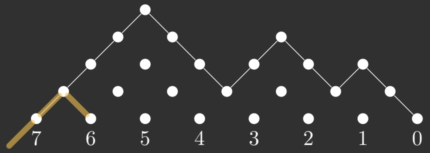

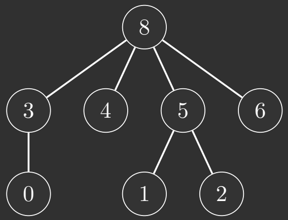

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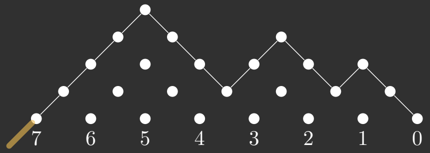

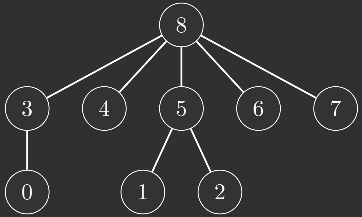

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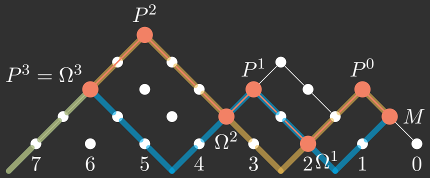

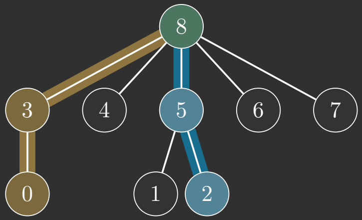

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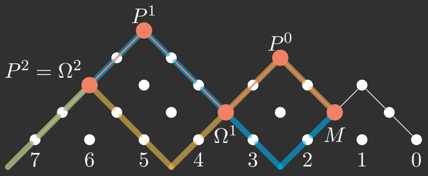
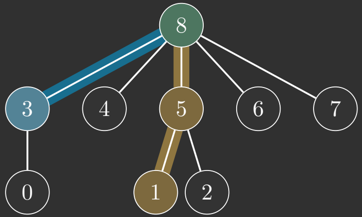
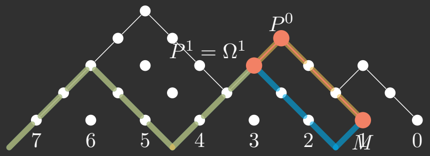
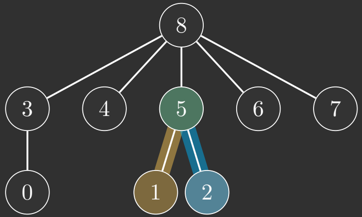
Toolbox: resolution tree





Toolbox: resolution tree





Stable Auslander algebras of Nakayama algebras

- You can generalise this process to work for a cyclic Nakayama algebras (type $\tilde{A}$), and therefore to *all* Nakayama algebras
- **Stable Auslander algebra:** $\underline{\mathrm{Aus}}(A) := \underline{\mathrm{End}}_A(M)$, $\quad M$ = additive generator of $\mathrm{mod}\, A$
- **Gorenstein dimension:** $\mathrm{gordim}\, A = \sup\{\, \mathrm{idim}\, P \mid P$ projective $\}$

Theorem (Giatagantzidis–Hartung–Marczinzik–Mustata–Rubey–Schmitz–J. '26+)

For a Nakayama algebra A,

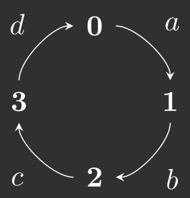
$$\mathrm{gordim}\, \underline{\mathrm{Aus}}(A) = \mathrm{gordim}\, \underline{\mathrm{Aus}}(A^{\mathrm{op}}).$$

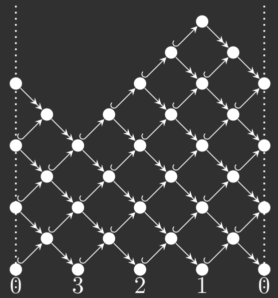
Thank you for listening!

- Klász, Marczinzik, Mellit, Rubey, Stump. *On the global dimension of Nakayama algebras and Dyck paths.* 2025.
- Giatagantzidis, Hartung, Marczinzik, Mustata, Rubey, Schmitz, J. *On the Gorenstein dimension of stable Auslander algebras for Nakayama algebras.* 2026+ (in preparation).
- Assem, Simson, Skowroński. *Elements of the Representation Theory of Associative Algebras, Vol. 1.* Cambridge University Press, 2006.
- Schiffler. *Quiver Representations.* Springer, 2014.

Cyclic Nakayama algebras

- A cyclic Nakayama algebra has a single oriented cycle as quiver
- The AR quiver lives on a **cylinder** (dotted lines are identified)





Kupisch series $c = [6, 9, 8, 7]$, cokupisch $d = [8, 9, 6, 7]$

Why Nakayama algebras are nice

- Uniserial: every module has a *unique* composition series — the submodule lattice is just a chain.
- Indecomposables are determined by two numbers: their top (which simple module generates them) and their length (the number of composition factors).
- The top of every indecomposable module is simple.
- Projective and injective resolutions are computed by a single walk on the AR quiver; there are no higher syzygies to chase separately.
- The AR quiver has a very clean shape.