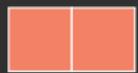
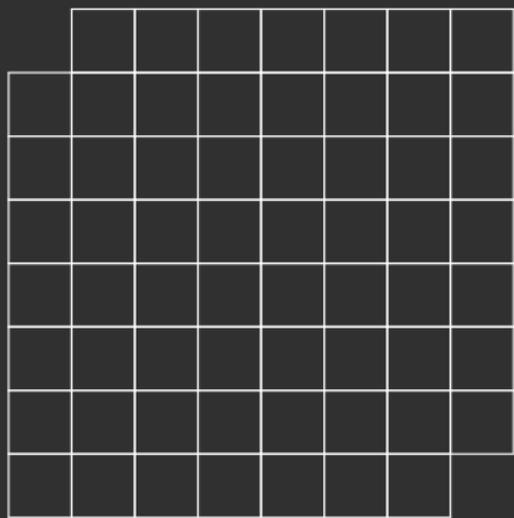


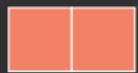
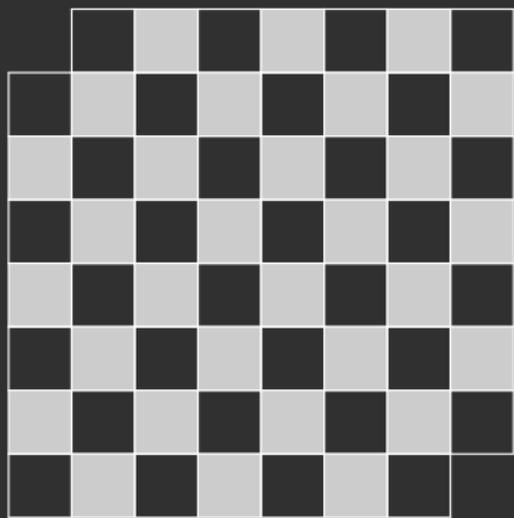
Tilings, 3-Colourings, and the Arctic Circle Phenomenon

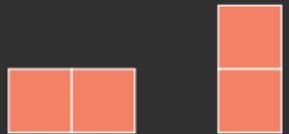
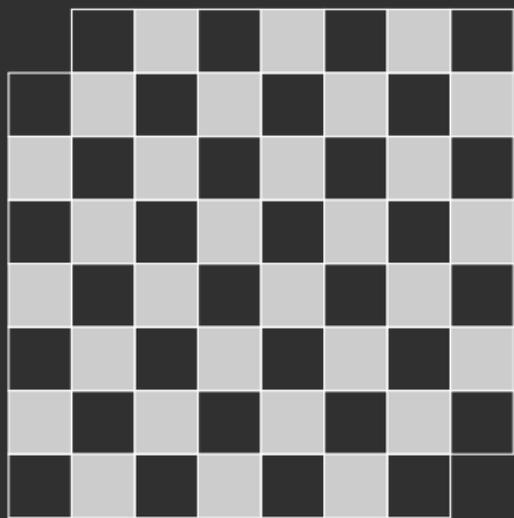
Nupur Jain

Oberseminar Combinatorial Synergies

A Simple Colouring Argument







30 white, 32 black



Conway introduced Tiling Groups,
and turned the problem of existence
of tilings into one of combinatorial
group theory



Thurston extended Conway's theory
and added a geometric flavour—in
particular, he defined height functions

Some questions we can ask about tilings

- 1 Does a tiling of a given region exist?
- 2 How many tilings of of the region exist?
- 3 Are there relations among the different tilings? Can we move from one tiling to another through a series of 'local moves'?  \leftrightarrow 
- 4 What does a 'typical' tiling of the region look like?
- 5 How hard is it to find a tiling?
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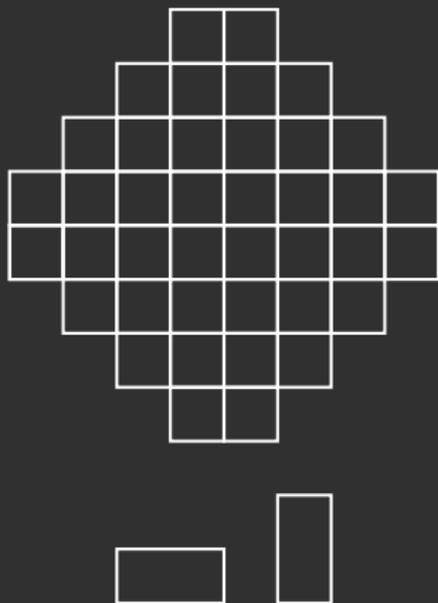
*Ref: Jim Propp's *A Pedestrian Approach to a Method of Conway, or, A Tale of Two Cities*

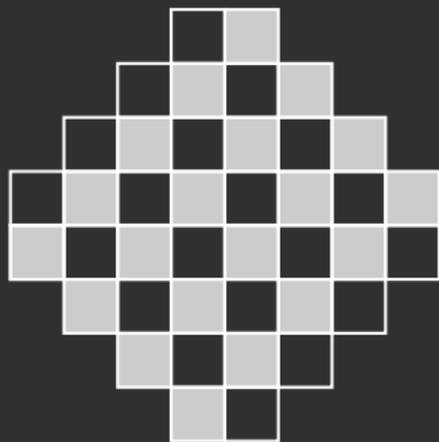
'Typical' Tilings

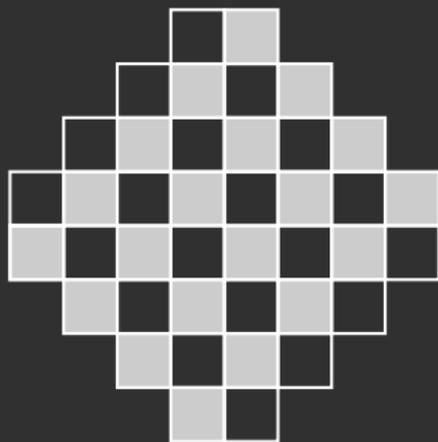
Uniform sampling

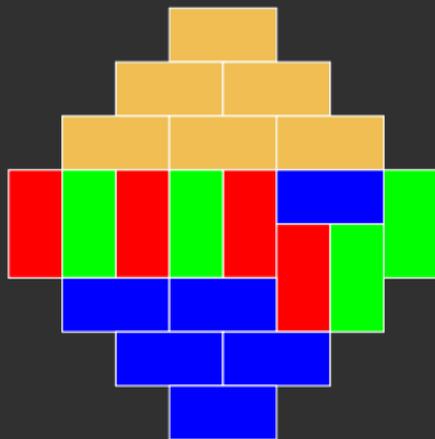
Suppose you are given a region R and a set of tiles. You generate all possible tilings of R with these tiles, and sample one randomly.

Are there any interesting features you would see with high probability?

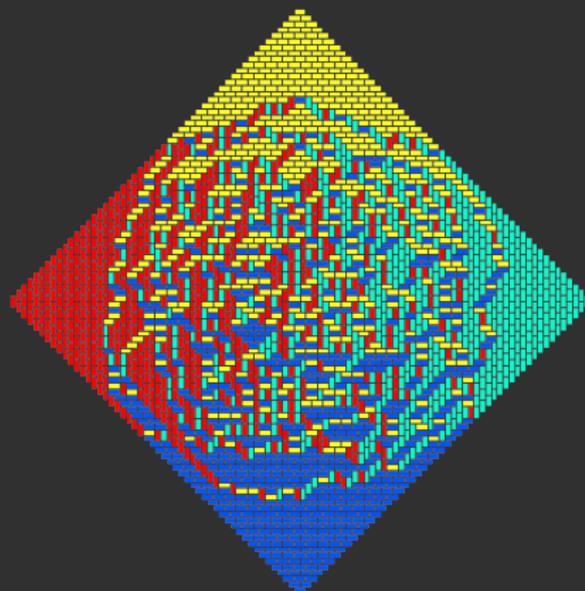




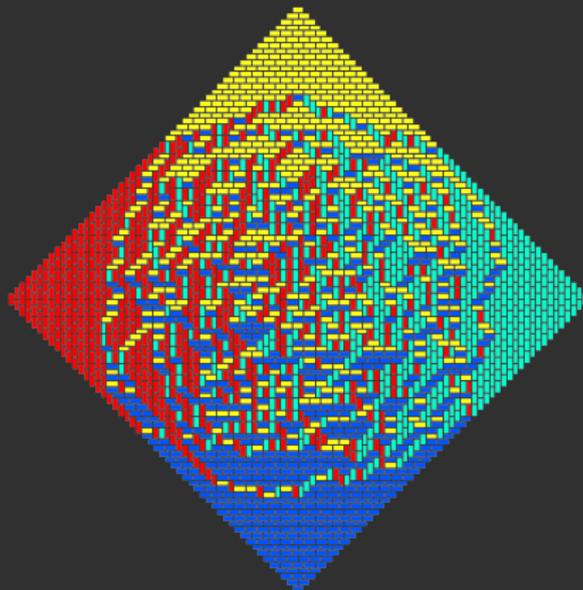




The Arctic Circle Phenomenon in 2D Dimer Tilings

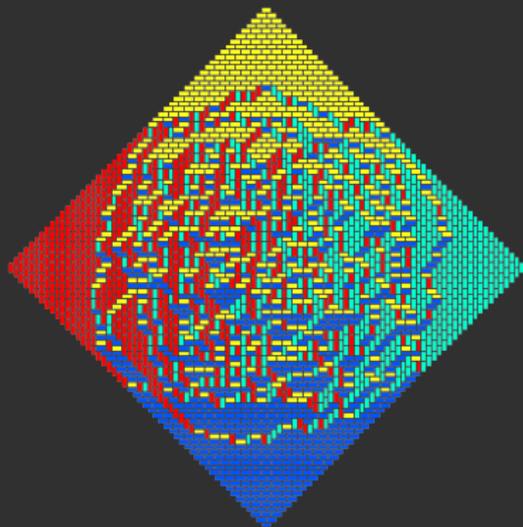


The Arctic Circle Phenomenon in 2D Dimer Tilings



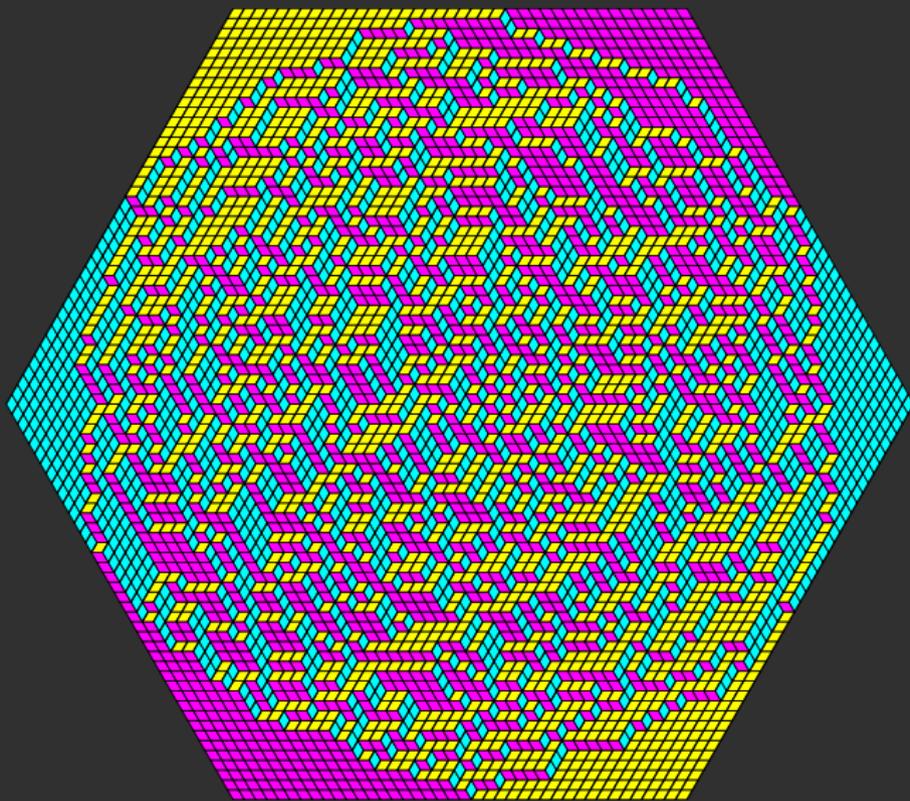
Jockusch, Propp, and Shor (1998) proved the Arctic Circle phenomenon for domino tilings of the Aztec diamond.

The Arctic Circle Phenomenon in 2D Dimer Tilings

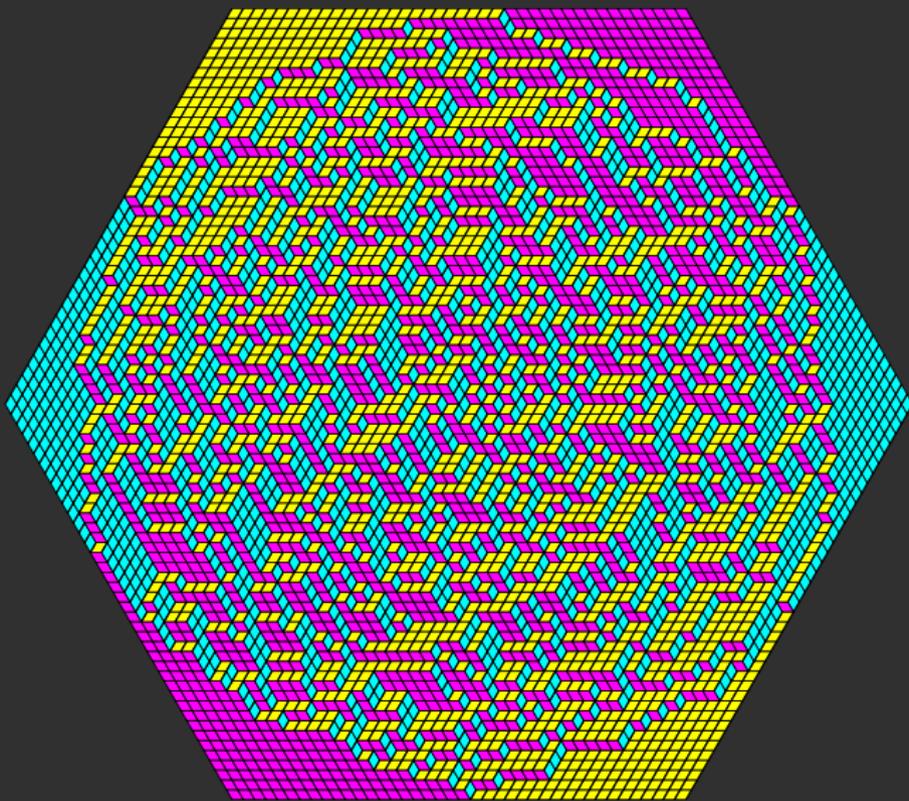


Theorem

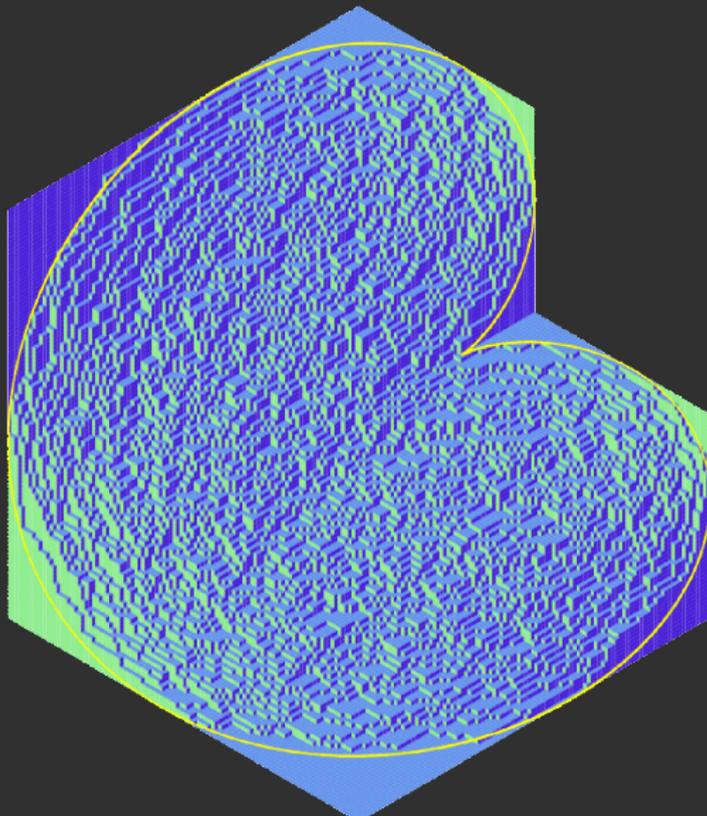
Fix $\epsilon > 0$. Then for all sufficiently large n , all but an ϵ -fraction of the domino tilings of the Aztec diamond of order n will have a temperate zone whose boundary stays uniformly within distance ϵn of the inscribed circle.



A randomly-sampled lozenge tiling



A randomly-sampled lozenge tiling



Another randomly-sampled lozenge tiling

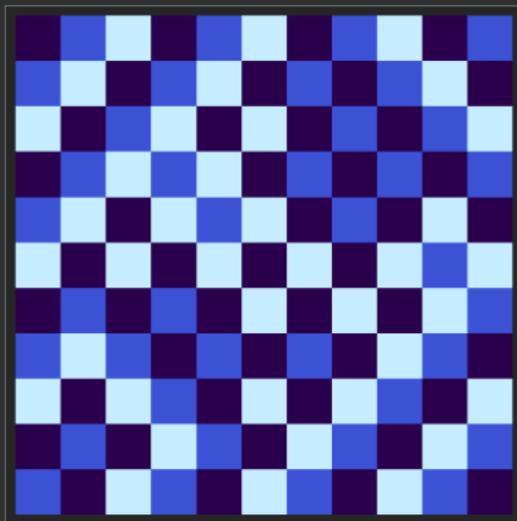
Generalising from dominoes

Instead of tiling with dominoes, which are 2×1 and 1×2 tiles, we can consider tilings using $n \times 1$ and $1 \times n$ tiles. These have been notably studied by Claire Kenyon and Richard Kenyon [KK92] and by Bodini and Rémila [B02].

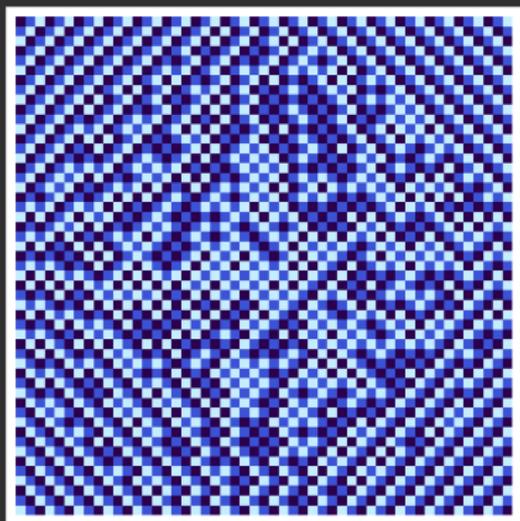
The Arctic Circle Phenomenon is conjectural in these cases.

Related: 3-colourings

A 3-colouring is an assignment of three colours to cells in a region so that adjacent cells do not have the same colour.



A random sample



A random sample of a 3-colouring of a 51×51 grid with a fixed boundary condition.

Amol Aggarwal (2020) proved the Arctic Circle phenomenon in this case.

Moving to 3 dimensions

Proving such results in 3 dimensions is much harder, since we lose a lot of machinery that is available in 2 dimensions.

In [CSW23], the authors (Nishant Chandgotia, Scott Sheffield, Catherine Wolfram) show a large deviations for domino tilings in 3 dimensions.

The Setup: 3-Colourings

Sampling: the naïve way

Say we want to sample a random 3-colouring. One way to do this is to generate all possible 3-colourings, and then picking one randomly.

The issue: the number of three colourings grows exponentially in the size of our region, so this is not feasible!

Markov chains

An alternative: use Markov chain sampling algorithms.

Definition (Markov Chain)

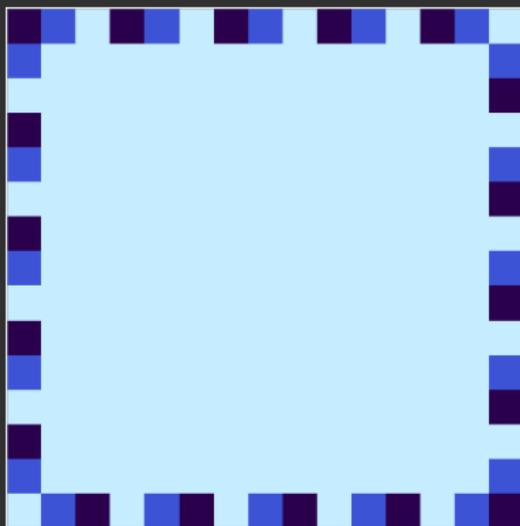
Given a state space \mathcal{X} and transition matrix P , a *Markov chain* is a sequence of random variables (X_0, X_1, \dots) satisfying the *Markov property*:

$$\begin{aligned}\Pr(X_{n+1} = y \mid X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) \\ = \Pr(X_{n+1} = y \mid X_n = x_n) = P(x, y).\end{aligned}$$

- Suppose the probability of moving from state x to state y is $P(x, y)$
- We can collect these probabilities in a matrix $P = (P(x, y))_{x, y \in \mathcal{X}}$
- An eigenvector of P is called the *stationary distribution* (denoted π) of the Markov chain if it has eigenvalue 1 and its entries sum to 1
- In “nice” cases, the stationary distribution is unique and the chain converges to it
- Our goal is to find a set of local moves on the space of 3-colourings such that the stationary distribution is the uniform distribution

Boundary conditions

Goal: find a random sample of a 3-colouring of a square grid whose boundary is fixed and looks like this:

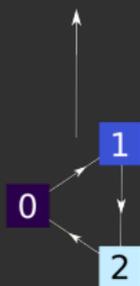


Lifting a colouring to a height function

We now use concepts introduced by Thurston to associate a 'height' to each cell in a 3-colouring.

We do this by 'lifting' the colouring from $\mathbb{Z}/3\mathbb{Z}$ to \mathbb{Z} .

... -3 -2 -1 0 1 2 3 4 5 6 7 8 ...

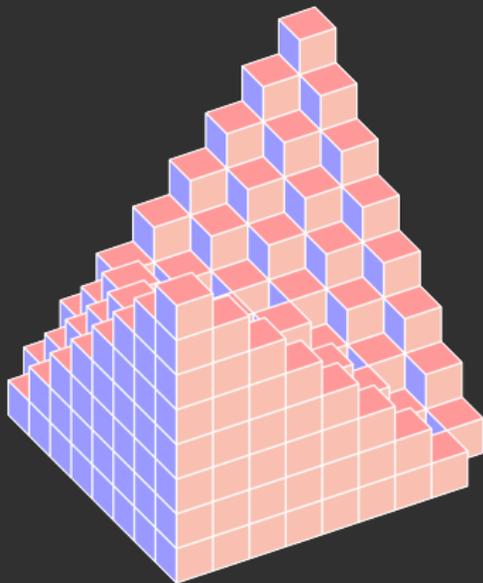


0	1	2	3	4	5	6	7	8
1	2	3	2	3	4	5	6	7
2	3	4	3	2	3	4	5	6
3	4	3	2	3	2	3	4	5
4	5	4	3	2	3	2	3	4
5	4	5	4	3	2	1	2	3
6	5	4	3	2	3	2	1	2
7	6	5	4	3	2	1	0	1
8	7	6	5	4	3	2	1	0

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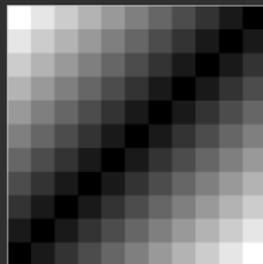
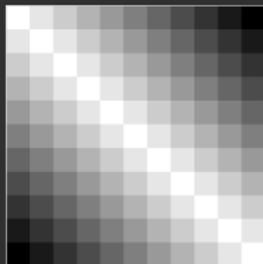
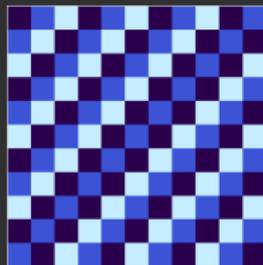
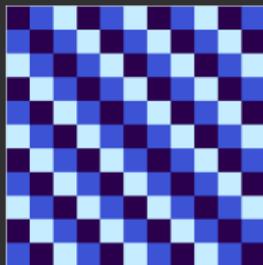
0	1	2	3	4	5	6	7	8
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Popping up and popping down



The height value at that cell changes by 2. If it increases by 2, we call this step 'popping up', otherwise we call it 'popping down'.

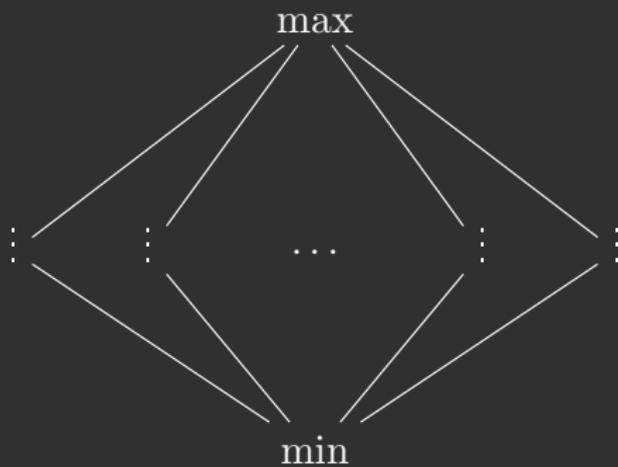
The partial order of colourings



The minimum colouring

The maximum colouring

The partial order of colourings



Local moves

- Turns out: you can move from any 3-colouring to any other 3-colouring through a series of pop-ups and pop-downs!
- This local move-connectedness property is very useful

The algorithm

Algorithm 1: A single step in the Markov chain

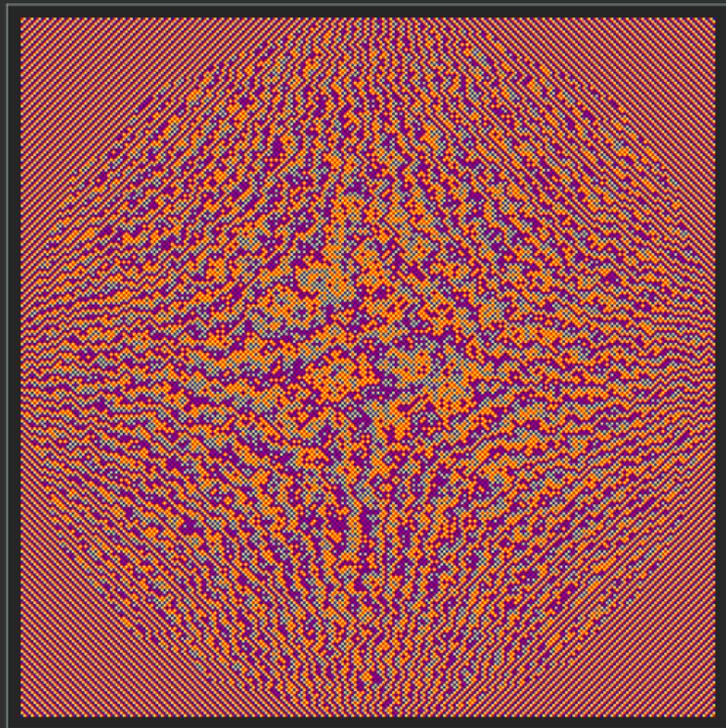
Data: 3-colouring of a grid

- 1 Pick a cell c uniformly randomly from the interior of the grid;
 - 2 Flip a fair coin to choose between popping up or popping down; let the outcome be P ;
 - 3 **if** operation P can be performed on cell c **then**
 - 4 | Perform operation P on cell c ;
 - 5 **end**
-

Sampling using the algorithm

- Start from any 3-colouring of the grid
- Perform the Markov chain step repeatedly
- After m steps, there will be a well-defined probability distribution on the space of 3-colouring of the grid
- It turns out: the limiting distribution is the uniform distribution!
- The longer we run the chain, the closer our result will be to a uniform sample
- Issues:
 - 1 We will only get an *approximately* uniform sample
 - 2 How long do we actually have to run the chain to get a good enough sample?

Sampling using the algorithm



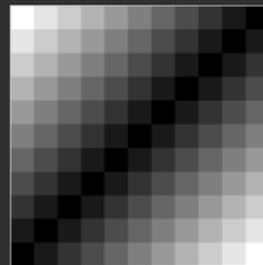
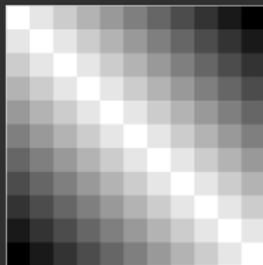
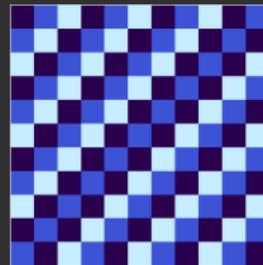
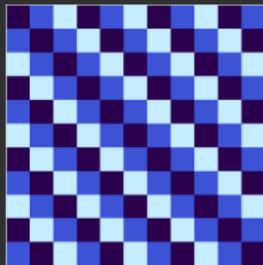
A 3-colouring of a 256×256 grid after 2.3 billion steps

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Coupling From the Past

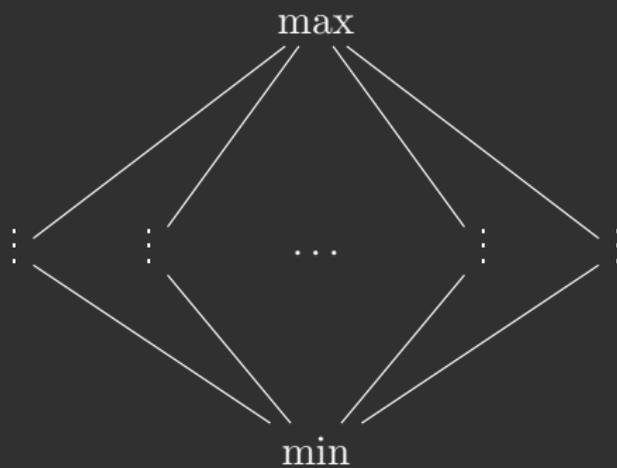
The partial order of colourings



The minimum colouring

The maximum colouring

The partial order of colourings



Enter: monotonicity

- Popping up/popping down is an order-preserving action!
 - This means if C_1 and C_2 are two 3-colourings of the grid with $C_1 \leq C_2$, then for any valid step s (as defined before), $s(C_1) \leq s(C_2)$
- **Upshot:** If we apply the same series of steps on C_1 and C_2 , then they may move closer in the poset or “collapse”, but they will not cross each other!
- In other words, the single step move “*squishes*” the poset

Coupling from the past

- Coupling from the past is a sampling algorithm invented by Propp and Wilson in 1996
- Advantages:
 - 1 Samples *perfectly* from the uniform distribution
 - 2 Decides runtime on its own
- Idea: Suppose the chain has been running from time $t = -\infty$. Then the colouring at time $t = 0$ is a uniform sample.

Coupling from the past

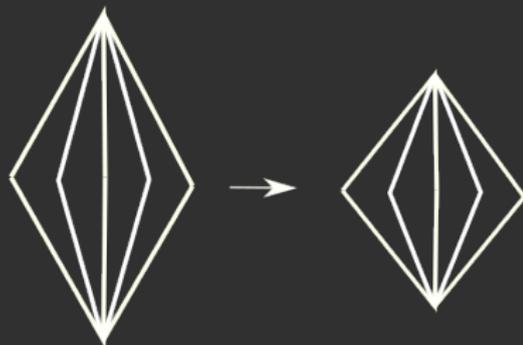
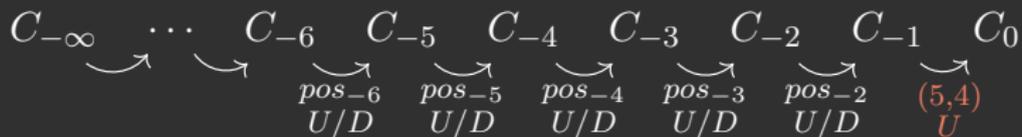
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The algorithm

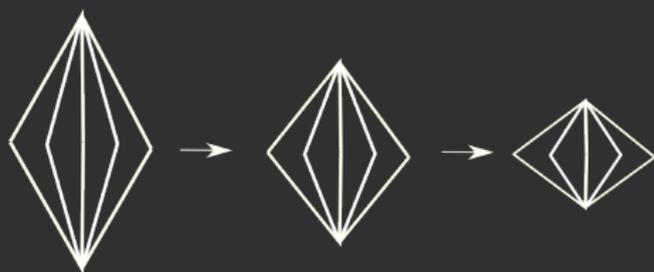
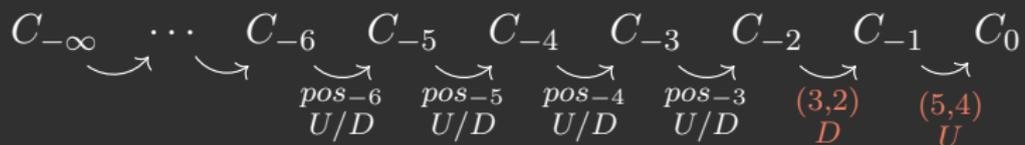
$$\begin{array}{cccccccc}
 C_{-\infty} & \cdots & C_{-6} & C_{-5} & C_{-4} & C_{-3} & C_{-2} & C_{-1} & C_0 \\
 \curvearrowright & \curvearrowright \\
 & & \text{pos}_{-6} & \text{pos}_{-5} & \text{pos}_{-4} & \text{pos}_{-3} & \text{pos}_{-2} & \text{pos}_{-1} & \\
 & & U/D & U/D & U/D & U/D & U/D & U/D &
 \end{array}$$



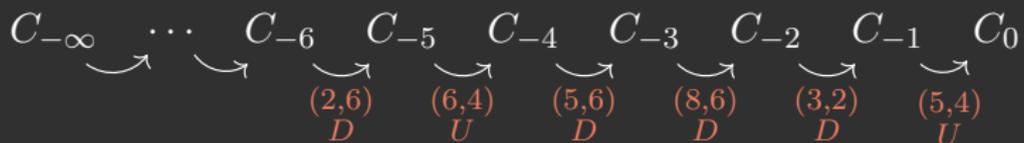
The algorithm



The algorithm



The algorithm



Exploiting the boundedness of the poset

Because of monotonicity, we only need to check if the min and max colourings have converged into a single colouring!

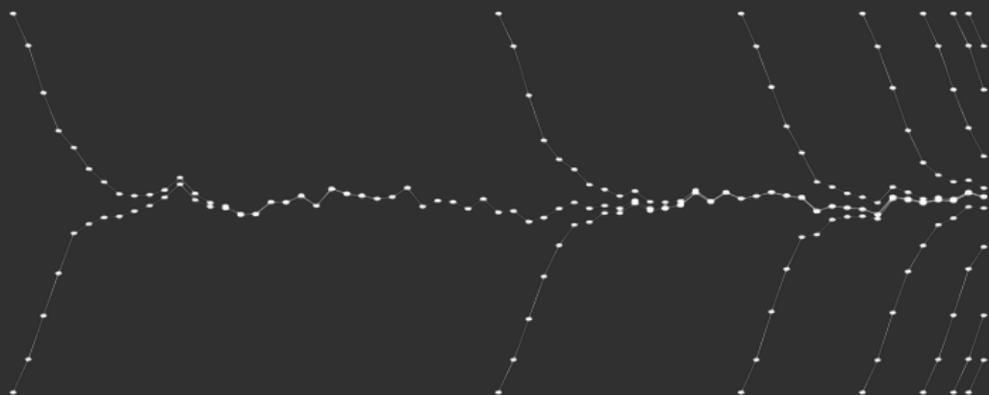


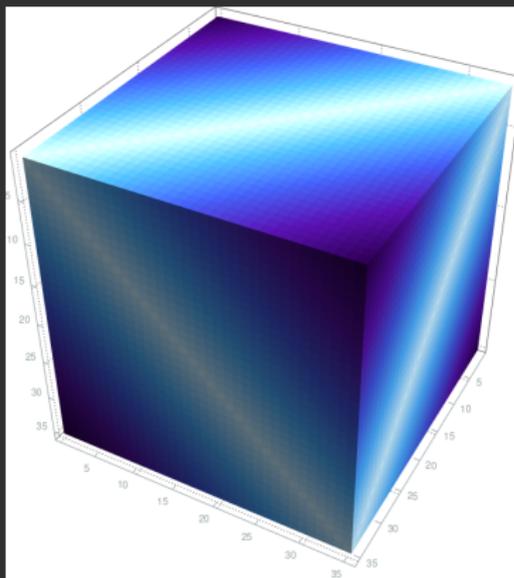
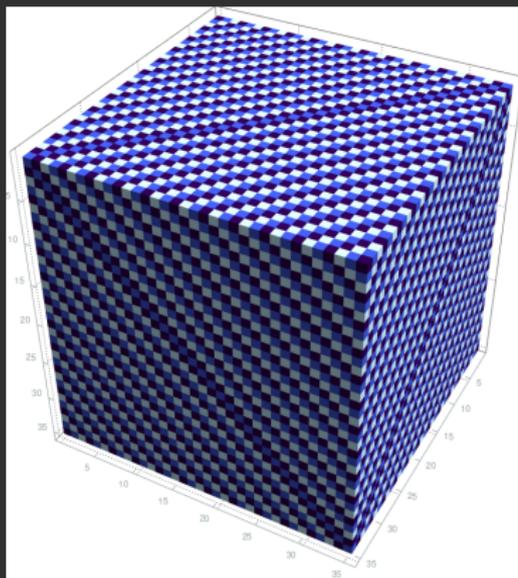
Illustration of Coupling From the Past showing the heights of the min and max height colourings starting at various starting points in the past [DAL09]

Outputs: a uniform sample of a 2-dimensional 3-colouring



A uniform sample of a 3-colouring of a 51×51 square grid. The left shows the actual 3-colouring, the center shows the height function, and the rightmost figure shows a plot of the height function.

Boundary conditions for 3 dimensions



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Thank You